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Investigating Two-Attribute Utility Function Regarding "No Independence Properties Hold"

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Abstract

In two-attribute statistical decision-making problems, two-attribute utility function is needed to be calculated for making the best decision. In the presence of any independence between the attributes, additive independence or mutual utility independence and so on, the calculation of two-attribute utility function is easy. But when no independence properties hold between the attributes, the calculation of twoattribute utility function will be difficult. In this paper two-attribute utility function is studied. There would be three main methods to calculate the two-attribute utility function regarding "no independence properties hold". These methods are: transformation of Attributes, Direct Assessment and Employment Utility Independent over Subsets of Consequences Space. These methods discussed completely with their deficiencies. Also we propose whitening technique to improve the transformation of attributes method.

Keywords: Statistical decision, Two-attribute utility function, Utility independence, Whitening technique.

1 | Introduction

During the last six decade, the utility theory is one of the main theories which are used for decision analysis in uncertainty condition. The preferences of the decision maker are quantified by this theory. The numbers that are allocated to each choice of the decision maker are the value of the choices. This number which called the utility, shows the priorities of the decision maker. The utility theory develops this number [1], [2].

Under multi-attribute utility theory, a decision maker formulates his or her preferences in terms of a scalar function, called a utility function, over the domain of attribute values. This function defines the decision maker's preferred attribute values and the tradeoffs between different attributes. Uncertainty is modeled probabilistically as a distribution over the possible attribute values for a given action on the part of a decision

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maker. For a properly-constructed utility function, the most preferred action of the decision maker corresponds to the action that maximizes expected utility [3], [4].

Multi-attribute utility analysis is utilized for resolving and assessing real-world decision making problems with several alternatives. Raiffa and Keeney [5] give comprehensive theoretical basis and deal with some applications of multi-attribute utility analysis. Multi-attribute utility analysis has been applied for real-world problems: evaluation of development policies of the municipal government, management of nuclear waste from power plants, economic analysis of South Korea, and especially, multi-criteria decision analysis for public and environment-related issues has attracted increasing attention [4].

The construction of a representative multi-attribute utility function is a fundamental step in decision analysis and has engendered a substantial literature. One of the widely used methods constructs the utility function by establishing the presence of certain utility independence conditions among the attributes to determine its functional form [5]. In particular, if there is mutual utility independence, then the multi-attribute utility function has either an additive or a multiplicative form [6], [7].

In this paper two-attribute utility function is studied. In case no independence properties hold between the attributes, there would be three main ways to calculate the two-attribute utility function. These methods are: Transformation of Attributes, Direct Assessment and Employment Utility Independent over Subsets of Consequences Space [5]. These methods discussed completely along with their deficiencies. Finally we proposed data whitening technique for improvement the transformation of attributes method. Data whitening changes the dependent data into independent ones, when the distribution of data is normal. Regarding that the consequences space of the most decision making problems has normal distribution, therefore, the proposed method can be applied in most of times.

2 | The Calculation of Single-Attribute Utility Function

In the following theorems let R be the consequence space

Theorem 1. Let $r, r_1, r_2 \in R$ is such that $r_1 \precsim r_2$ and $r_1 \precsim r \precsim r_2$. Then there exists a number α (0 ≤ α ≤ 1) such that:

$$r \sim ar_2 + (1-a)r_1. \quad \textbf{(1)}$$

Theorem 2. Let $r_1, r_1, r_2 \in R$ is such that for some value of α (0 ≤ α ≤ 1), $r_2 \sim ar_3 + (1-a)r_1$.

Than

$$U(r_{2)} = aU(r_3) + (1-a)U(r_1). \quad \textbf{(2)}$$

To calculate single-attribute utility function, first two consequences of r0 and r1 are selected such that the utility of r0 for the decision maker is equal to zero and the utility of r1 is equal to 1 (if possible, r0 is selected as the worst consequence and r1 as the best consequence). Let r ∈ R is such that $r_0 \precsim r \precsim r_1$; to calculate the utility of r, decision maker is required to state the p so that the two alternatives have the same value for him/her. In other words, she/he is indifferent about the two following alternatives [8].

I. Receive r with probability of 1 (Certainty).
II. Participating in a gambling, in which with the probability of p the consequence of r_1 and with the probability of $1-p$ the consequence of r_0 will be obtained.

These two alternatives are shown in the following:

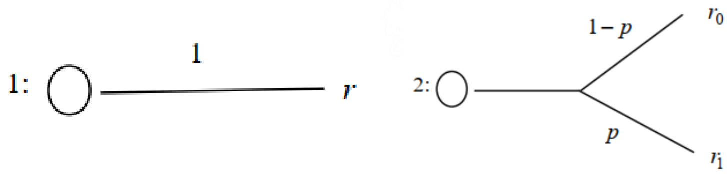

Fig. 1. Single characteristic utility function.

According to *Theorems 1* and *2* we have:

$$r \sim (1-p)r_0 + pr_1, \quad (3)$$

and

$$u(r) = (1-p)u(r_0) + pu(r_1). \quad (4)$$

Also according to the definition of r_0 and r_1 we have: $u(r_0) = 0 \& u(r_1) = 1$. So we have:

$$u(r) = (1-p) \times 0 + p \times 1 = p. \quad (5)$$

As it was explained in the theorem 1, the above-mentioned probability function exists and is unique (It must be explained that if $r \succ r_1$ or $r \prec r_0$ we can act similar to the illustrated method).

3 | Two-Attribute Utility Function

Assume that the consequence space of X includes two attributes of Y and Z, it means that for every $x \in X$ we have: $x = (y, z); y \in Y$ and $z \in Z$, so the consequences space, is called twoattribute space. If the decision making problem has two attributes, utility function defined on this space, is referred to two-attribute utility function. In other words, if the attributes of the decision-making problem are X and Y, the two-attribute utility function of $U(x, y)$ is defined from the space of $X \times Y$ into the set of real numbers.

3.1 | Utility Independence

Assume that the consequences space includes two attributes of X and Y, if the decision maker recognizes his/her preference for some value of X does not depend on the value of Y, thus the attributes of X is utility independence from the attribute of Y. Utility independence enables us to investigate the utility function of the X attribute independently form the Y attributes [9].

Here, another definition of utility independence is expressed, which is equal to the previous definition.

Definition 1. Attribute X, is called the utility independence from the attribute Y whenever conditional preference of x for the constant value of y = y0 does not depend on the value of y0 . In other words, for each y1 , y0 Y We have: u(x y = y0) = u(x y = y1) . This independence is shown as: X(UI) Y.

3.2 | Mutual Utility Independence and Additive Independence

Mutual utility independence for two attributes X1 and X2 is true when the attribute X1 is utility independence of X2 and the attribute X2 is utility independence of X1. In this case the appropriate model to display the two-attribute utility function is the multi-linear model which its formula is as follow:

$$u(x_1, x_2) = w_1 u_1(x_1) + w_2 u_2(x_2) + w_{12} u_1(x_1) u_2(x_2), \quad (6)$$

where ui(xi): the single-attribute utility function. i = 1, 2.

wi: is the weight or the importance of attribute Xi, that its value for each i is between zero and one.

w12: represents the mutual effects between the attributes X1 and X2. It must be noted that: w1+w2+w12=1.

The great benefit of mutual utility independence, if it exists, is that it enables the decision maker to concentrate initially on deriving utility function for one attribute at a time without the need to worry about the other attributes. If this independence does not exist then the analysis of decision making problem can be extremely complex [5]. Attributes X_1 and X_2 have additive independence if and only if

$$u(x_1, x_2) = w_1 u_1(x_1) + w_2 u_2(x_2). \quad (7)$$

This model is known as additive model, it is clear that when X1 and X2 have additive independence, they have mutual utility independence also. But the converse of this case is not necessarily true [10].

3.3 | Two-Attribute Utility Function with One Utility-Independent Attribute

In the previous sections, the formula of two-attribute utility function, when mutual utility independence or additive independence exists between two attributes of decision making problem, was explained. Here, the formula for two- attribute utility function will be stated, when only one of the attributes has the utility independence of the other one.

Theorem 3. If attribute X is the utility independent of the attribute Y, then

$$u(x, y) = u(x_0, y)\big(1 - u(x, y_0)\big) + u(x_1, y)u(x, y_0), \quad (8)$$

where u(x, y) is normalized by $u(x_0, y_0) = 0$ and $u(x_1, y_0) = 1$ [5].

4 | What to do if "No Independence Properties Hold"

In the previous sections, when there was a kind of utility independence between the attributes, a formula was exposed to calculate u(x, y) according to the type of independence. The question which is concerned here is, how u(x, y) can be calculated when no independence properties hold between the two attributes. To solve this problem three main methods have been provided. These methods discussed completely in this section with their deficiencies.

4.1 | Transformation of Attributes

In this method, the original attributes are changed into the new attributes in order that the new attributes are independent. In other words, if X and Y are two attributes of the main problem, then two attributes of S = f (X ,Y) and T = g(X ,Y) are defined so that a type of utility independence exists between S and T. in this case first the two-attribute utility function of S and T could be calculated. Then the utility function of X and Y could be extracted from u(s,t). This method is explained with this example.

Example 1. assume that no utility independence properties exist among the original attributes X and Y. Still it may be possible to define new attributes S ≡Y + X and T ≡ Y - X that do possess independence properties. For instance, S and T might be additive independence with the form

$$u(s, t) = s^2 + t. \quad (9)$$

In this case, the utility function of original attributes X and Y will be as follow:

$$u^*(x, y) = u(s(x, y), t(x, y)) = (x + y)^2 + (y - x) = y^2 + y + x^2 - x + 2xy. \quad (10)$$

4.2 | Direct Assessment

In this method u(x, y) is calculated directly, like the single-attribute utility function, which was investigated before.

4.3 | Employment Utility Independent Over Subsets of X×Y

This idea is simple. In this method it is enough to subdivide the consequence space to parts such that there is a kind of utility independence in each of these parts. In other words, assume that A is the consequence space of the intended decision making problem. In this method A is subdivided into the parts of $A_1, A_2, \dots, A_n (A = \cup_n^{\pi} A_i; A_1 \neq \phi$ for all i) such that in each A_i there is a type of utility independence among the attributes. According to the independence in each A_i, the utility function is calculated in this space and finally by combining them the utility function is obtained for the all Space of A. This method is explained more with an example.

Example 2. Suppose that we want to assess $u(x, y)$ over the space $X \times Y$, which is shown in the following figure ($y' \leq y \leq y''$ and $x' \leq x \leq x''$) where preference are increasing in both attribute. For $y \leq y_0$, attribute X is utility independent of attribute Y, so from 8 , if we set $u_1(x', y_0) = 0$ and $u_1(x'', y_0) = 1$, then

$$u_1(x, y) = u_1(x', y)\big(1 - u_1(x, y_0)\big) + u_1(x'', y)u_1(x, y_0); y \leq y_0,\ x' \leq x \leq x''. \quad (11)$$

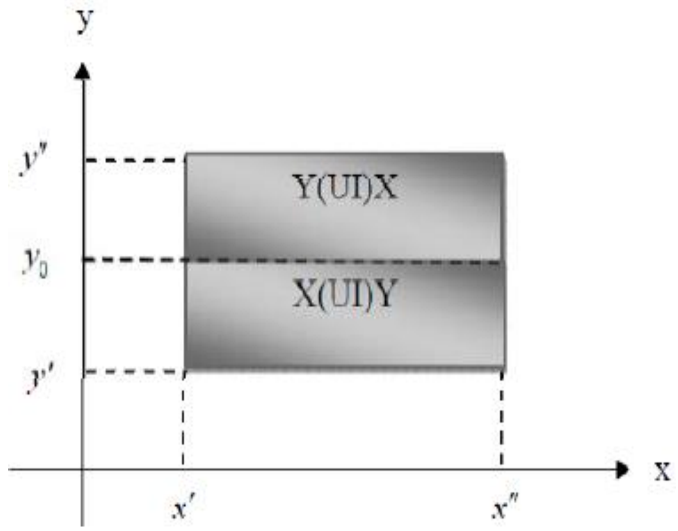

Fig. 2. consequences space.

For the rest of the original region, suppose Y is utility independent of X, so if we set $u_2(x', y'') = 1$ and $u_2(x', y_0) = 0$, then

$$u_2(x, y) = u_2(x, y_0)\big(1 - u_2(x', y)\big) + u_2(x, y'')u_2(x', y)\ ; y_0 \leq y,\ x' \leq x \leq x'' \quad (12)$$

Since both u_1 and u_2 have the same origin, then in order to consistently scale u_1 and u_2 we need only determine a scaling constant λ define by

$$\lambda = \frac{u_2(x'', y_0)}{u_1(x'', y_0)}. \quad (13)$$

In this case a consistent utility function for all $X \times Y$ is

$$u(x, y) = \begin{cases} \lambda u_1(x, y), & y \leq y_0, x' \leq x \leq x'', \\ u_2(x, y), & y_0 \leq y, x' \leq x \leq x''. \end{cases} \quad (14)$$

5 | Defectives of Provided Methods to Calculate Two-Attribute Utility function with respect to “no Independence Properties Hold”

This section deals with investigating the disadvantages of the methods which have been provided in the previous section to calculate two-attribute utility function whereas no independence properties hold.

5.1 | Transformation of Attributes

The most important part of this method is how to select the functions f and g, so that S and T have the type of utility independent. The first problem of this method is that, the way of selecting S and T has not been mentioned at all. In fact, how the functions f and g must be selected so that S and T are independent form each other, has been remained without any answer in this method. The second problem of this method is significant or interpretable of S and T. For example, assume that the attribute X is the duration of a project and Y is its cost, and T is defined as $T = X^2 - Y^2$. Then, what is the attribute T must be interpreted so that its utility function can be calculated.

5.2 | Direct Assessment

As explained before, the utility of the members of the consequences space set are calculated directly. For two-attribute state, this method is as follow: first, two consequences of (x_0, y_0) and (x_1, y_1) respectively with utilities of zero and one are selected. Then, to calculate the utility of (x, y), the decision maker is asked to state the quantity of p so that she/he is indifferent about the two following alternatives.

I. Receive (x, y) with probability of 1. (Certainty).

II. Participating in a gambling in which with the probability of p the consequence of (x_1, y_1) and with the probability of 1 - p the consequence of (x_0, y_0) will be obtained.

Like single-attribute method, p is the utility of (x, y). According to the consequences being two-attribute, comparing the two alternatives and selecting the quantity of p is difficult for the decision maker. Finally, the decision maker states the quantity of p so that, it does not adapt with his/her preferences and the result of decision making would be incorrect. Therefore, this method in practically is very difficult and full of mistakes to use.

5.3 | Employment Utility Independent over Subsets of X×Y

As it was explained, in this method, the consequences space is divided into subsets such that there is a kind of utility independence in these subsets. In this method nothing has been mentioned about the way of recognizing and selecting these subsets, although this part of the method is the most important part of it. Therefore, this method does not have the efficiency to calculate the two-attribute utility function as the previous methods.

According to the mentioned problems, it is observed that the methods provided for calculating the two-attribute utility function with respect to no independence properties hold between the attributes, are not efficient. In continue, we propose whitening technique to improve the transformation of attributes method.

6 | Whitening

In whitening, dependent variables are transformed by a linear transformation into a set of new variables, which are uncorrelated. In other words, if the whitened variables are shown by $\tilde{X}$, covariance matrix of $\tilde{X}$ equals the identity matrix:

$$E(\tilde{X}\tilde{X}^7) = I. \tag{15}$$

The whitening transformation is always possible. One popular method for whitening is to use the Eigen-Value Decomposition (EVD) of the covariance matrix $E(XX^T)=EDE^T$, where E is the orthogonal matrix of eigenvectors of $E(XX^T)$ and D is the diagonal matrix of its eigenvalues,

$D = diag(\lambda_1, \lambda_2, ..., \lambda_n)$. Whitening can now be done by:

$$X = ED^{\frac{-1}{2}}E^{\tau}X. \tag{16}$$

It is easy to check that now $E(\chi_{X\tilde{X}^{\tau}}{}^{\tau}) = I$ [11].

7 | Proposing Whitening Technique for Improvement the Transformation of Attributes Method

As mentioned in the previous section, by using whitening technique, correlated variables always can be transformed into the new variables which are uncorrelated. Also if the original variables have bivariate normal distribution, then new variables will be independent (in bivariate normal distribution, the variables are independent if and only if they are uncorrelated) [12].

In many decision making problems, consequences space follow normal distribution. According

to the stated issues, it is concluded that by using whitening technique and transformation of

attributes, which are independent. In other words, new attributes may be have mutual utility independence that for confidently it must be checked. In other terms, in transformation of attributes method, f and g could be selected as linear transformations resulted from data whitening. According to the stated issues, by this transformation, new attributes may be shall independent. Also, regarding that g and f are linear functions, the utility function of the original attributes can be extracted easily from the utility function of new attributes.

8 | Conclusions

Often, consequences of a decision making problems are uncertain. Therefore, to obtain the best alternative, it is necessary to calculate utility function and select the alternative which has the most expected utility. When, the space of consequences includes just one attribute, the calculation of the utility function is simple that is explained in this paper. When consequences space includes two independent attributes, two-attribute utility function can be obtained by calculating single-attribute utility function for every attribute and their combination. But as explained before, if there is no kind of utility independence between the attributes, the calculation of the two-attribute utility function was very difficult. In this case, there are three main methods to calculate the two-attribute utility function.

These methods are: transformation of attributes, direct assessment and employment utility independent over subsets of consequences space. These methods discussed completely with their deficiencies. Finally we proposed data whitening technique for improvement the transformation of attributes method. Data whitening changes the dependent data into independent ones, when the distribution of data is normal. Regarding that the consequences space of the most decision making problems has normal distribution, therefore, the proposed method can be applied in most of times.

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Author Contribution

Bohlol Ebrahimi: conceptualization, methodology, drafting the original manuscript, and data analysis.
Majid Aminnayeri: supervision, review, and editing. Reza Ramazani: data collection, validation, and writing—review and editing.
All authors have read and approved the final version of the manuscript.

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Conflicts of Interest

The authors declare that there are no conflicts of interest related to this research.

Data Availability

The data used and analyzed in this study will be available upon reasonable request from the corresponding author.

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